

The two flavour Schwinger model: scaling of the scalar condensate

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We investigate the continuum limit scaling of the scalar condensate in the $N_f = 2$ Schwinger model on the lattice. We employ maximally twisted mass Wilson fermions and overlap fermions. We compute the scalar condensate by taking the trace of the propagator (direct method) and by utilizing the integrated Ward-Takahashi identity. While the scalar condensate comes out consistent using these two methods for a given kind of lattice fermions, we find –quite surprisingly– large discrepancies for the scalar condensate between twisted mass and overlap fermions. These discrepancies are only resolved when using the point split current for twisted mass fermions.

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1. Introduction

The Schwinger model [1] is a good test ground for 4-dimensional QCD due to the properties of asymptotic freedom and the existence of non-perturbatively generated bound states. In refs. [2] we have studied on a lattice the scaling properties of meson masses for a number of different fermion discretizations using Wilson, hypercube, Wilson twisted mass (TM) and overlap (OV) fermions. In these investigations, we found that the pseudo scalar mass scales with an $O(a^2)$ behaviour for all fermion discretizations. The same $O(a^2)$ scaling behaviour was also observed in [3] in a finite volume scaling analysis in the Schwinger model. These findings can be attributed to the super-renormalizability of the model.

The aim of the present work is to perform another scaling test of the scalar condensate as an additional, non-trivial quantity. In the case of $N_f = 1$, ref. [4] shows that the continuum and the chiral limit come out to be consistent with the predictions in the continuum theories [5, 6]. Here we investigate the scaling toward the continuum limit in the $N_f = 2$ Schwinger model using maximally twisted mass fermions and overlap fermions. Emphasis will be put on the comparison of different methods and definitions of the scalar condensate used for its computation. The methods we apply and which will be detailed below are the direct calculation through the trace of the (inverse) Dirac operator employed and the integrated axial Ward–Takahashi identity. As a quite surprising outcome of this investigation we find that a naive approach of computing the scalar condensate by using the local definition, $\langle \bar{\psi}(x)\psi(x) \rangle$, does not lead to a consistent continuum limit.

Another aspect of our work which was presented in the poster has been the investigation of the question of how many eigenmodes of the used lattice Dirac operator are needed to approximate the pseudo scalar correlator to a certain precision. In particular, we have studied how this number of eigenmodes scales towards the continuum limit in the case of overlap fermions. We can, for lack of space, not discuss this issue in this proceedings write-up and refer to ref. [7] for a detailed discussion.

2. Lattice actions and calculation methods

We have employed maximally twisted mass Wilson fermions and overlap fermions as chirally improved and chirally invariant formulations of lattice fermions, respectively, in this work. Since both of these kind of lattice fermions are $O(a)$ improved, we expect only $O(a^2)$ lattice artefacts.

The Neuberger operator [8] as a realization of overlap fermions is given as

$$D_{\text{ov}} = \left(1 - \frac{m_q a}{2}\right) D_0 + m_q \quad , \quad D_0 = \frac{1}{a} \left[1 + \frac{D_{\text{kernel}}}{\sqrt{D_{\text{kernel}}^\dagger D_{\text{kernel}}}} \right] . \quad (2.1)$$

In the following, we will use as kernel the hypercube operator [9], i.e. $D_{\text{kernel}} = D_{\text{hyp}}$ with parameters obtained from optimizing scaling. This lattice fermion has an exact (lattice) chiral symmetry due to the Ginsparg–Wilson relation [10].

For Wilson twisted mass fermions, the lattice Dirac operator is given by

$$D_{\text{tm}}(x, y) = (m_0 + 2)\delta_{x,y} - \frac{1}{2} \sum_{\mu=1}^2 [(1 - \sigma_\mu)U_\mu(x)\delta_{x,y-\hat{\mu}} + (1 + \sigma_\mu)U_\mu^\dagger(y)\delta_{x,y+\hat{\mu}}] + i\mu_{\text{tm}}\sigma_3\tau_3\delta_{x,y} . \quad (2.2)$$

We use the standard Pauli-matrices σ_μ ($\mu = 1, 2, 3$) with $\sigma_3 = \text{diag}(1, -1)$ and denote with $\hat{\mu}$ the unit vector shift in direction μ . The parameters m_0 and μ_{tm} denote the untwisted and twisted bare fermion masses, respectively. When m_0 is tuned to a critical value, $m_0 = m_{0,\text{crit}}$, by tuning the PCAC quark mass to zero, we reach maximally twisted mass fermions [11]. In this case, the theory is automatic $O(a)$ -improved and physical (parity even) quantities scale with an $O(a^2)$ behaviour toward the continuum limit.

There are two methods on the lattice, the direct and the integrated Ward–Takahashi identity methods, to calculate the scalar condensate which we will use for overlap and maximally twisted mass fermions.

The direct method is obtained from the trace of the fermion propagator with gauge backgrounds. The direct method is defined as follows for the overlap and twisted mass fermions respectively;

$$\Sigma_{\text{direct}}^{\text{ov}} = \frac{1}{V} \sum_x \text{Tr} \left[\left(1 - \frac{aD_0}{2} \right) D_{\text{ov}}^{-1} \right]_{(x,x)} , \quad \Sigma_{\text{direct}}^{\text{tm}} = \frac{1}{V} \sum_x \text{Tr} [i\sigma_3 \tau_3 D_{\text{tm}}^{-1}]_{(x,x)} . \quad (2.3)$$

For Wilson fermion, there is a term proportional to $\frac{1}{a}$ in 2-dimensions due to the explicit breaking of chiral symmetry [12]. However overlap fermions and maximally twisted mass fermions do not have such a problem and thus no $\frac{1}{a}$ term in 2-dimensions [2] appears. The same is true for the $\frac{1}{a^2}$ term in 4-dimensions.

Next we introduce the integrated Ward–Takahashi identity method [12] which reads

$$\Sigma_{\text{iWT}}^{\text{ov}} = 2m_q \sum_x \langle P^+(x) P^-(0) \rangle , \quad \Sigma_{\text{iWT}}^{\text{tm}} = 2\mu_{\text{tm}} \sum_x \langle P^+(x) P^-(0) \rangle . \quad (2.4)$$

$\Sigma_{\text{iWT}}^{\text{ov}}$ is obtained from the PCAC relation with the operator $P^\pm(x) = \bar{\psi}(x) \sigma_3 \tau_\pm [(1 - \frac{aD_0}{2}) \psi](x)$ and $\Sigma_{\text{iWT}}^{\text{tm}}$ from PCVC relation for TM fermions; $\partial_\nu^* \langle V_\nu^+(x) P^-(0) \rangle = 2\mu_{\text{tm}} \langle P^+(x) P^-(0) \rangle - \delta_{x,0} \langle S_0(x) \rangle$ where $P^\pm(x) = \bar{\psi}(x) \sigma_3 \tau_\pm \psi(x)$, $S_0(x) = i\bar{\psi}(x) \sigma_3 \tau_3 \psi(x)$.

3. Calculating the scalar condensate

We have carried out numerical simulations in the following setting. The lattice size is 20×20 and the statistics is over 1000 thermalized and statistically independent configurations. The quark mass is fixed as $z = \left(m_q \sqrt{\beta} \right)^{2/3} = 0.4$. The error estimate is done by the method of ref. [13]. The gauge action (S_G) is the Wilson plaquette action with the dimensionless coupling constant $\beta = \frac{1}{e^2 a^2}$.

To obtain results for dynamical fermions, for OV fermions, we simulated only the gauge action and used the re-weighting method and the spectral representation from eigenvalues and eigenmodes to compute physical observables: $\langle \mathcal{O} \rangle_{\text{unquench}} = \frac{\langle \det^{N_f}(D_f) \cdot \mathcal{O} \rangle_{S_G}}{\langle \det^{N_f}(D_f) \rangle_{S_G}}$. For TM fermions, we also generated configurations for the full action using the HMC algorithm.

We want to remark first that all results obtained by using either $\Sigma_{\text{direct}}^{\text{ov}}$ ($\Sigma_{\text{direct}}^{\text{tm}}$) or $\Sigma_{\text{iWT}}^{\text{ov}}$ ($\Sigma_{\text{iWT}}^{\text{tm}}$) came out to be completely consistent when considered for each kind of lattice fermion separately. Therefore, we discuss in the following only one of these cases. From a dimensional analysis, we expect a logarithmic term in β , [4, 2]

$$\sqrt{\beta} \Sigma = A + B/\beta + C \log(\beta) . \quad (3.1)$$

Because of universality and super-renormalisability, the coefficient multiplying the logarithmic term is universal and can be evaluated as $C = \frac{m_q \sqrt{\beta}}{2\pi}$. This allows us to define a subtracted scalar condensate,

$$\sqrt{\beta} \Sigma_{\text{sub}} = \sqrt{\beta} \Sigma - C \log(\beta) . \quad (3.2)$$

Fig. 1 shows $\Sigma \sqrt{\beta}$ and $\Sigma_{\text{sub}} \sqrt{\beta}$ in the case of OV fermions. When the logarithmic term is subtracted, we observe a perfectly linear behaviour of $\Sigma_{\text{sub}} \sqrt{\beta}$ as a function of $1/\beta$.

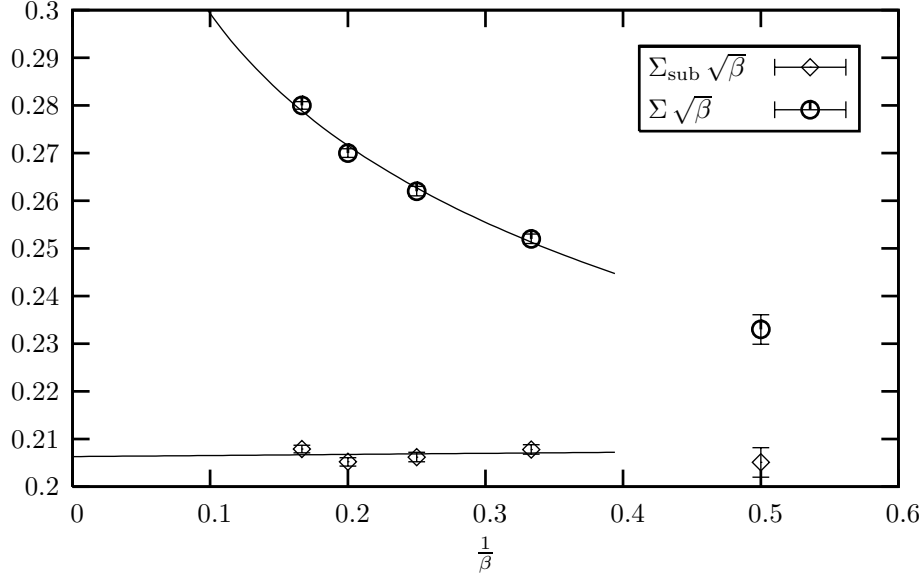


Figure 1: $\sqrt{\beta} \Sigma$ and $\sqrt{\beta} \Sigma_{\text{sub}}$ in the case of OV fermions as a function of $1/\beta$ for a fixed value of $z = 0.4$.

In order to tune to maximal twist we have determined the critical Wilson mass in the pure Wilson theory, i.e. setting $\mu_{\text{tm}} = 0$, by tuning the (untwisted) PCAC quark mass to zero. The values of the $m_{0,\text{crit}}$ can be found in ref. [2].

In fig. 2, we compare the (direct) TM fermion scalar condensate with the corresponding (direct) OV fermion data. The important result is that $A_{\text{tm}} \neq A_{\text{ov}}$, see eq. (3.1). Thus the continuum extrapolated values of the scalar condensate from both discretizations differ when the fitting function of eq. (3.1) is used.

In order to shed some light on the discrepancy between the results for TM and OV fermions discussed above, we suggest to use an improved current for TM fermions, namely the 1-point splitting current, given as

$$S_{\text{tm}}^{\text{imp}}(x) = \frac{1}{d} \sum_{\mu=1}^d \frac{1}{2} [\bar{\psi}(x) \gamma_5 \tau_3 U_{\mu}(x) \psi(x + \hat{\mu}) + \bar{\psi}(x) \gamma_5 \tau_3 U_{\mu}^{\dagger}(x - \hat{\mu}) \psi(x - \hat{\mu})] \quad (3.3)$$

for the direct method. For the integrated Ward–Takahashi identity we find

$$\partial_v^* \langle V_v^+(x) \hat{P}^-(0) \rangle = 2\mu_{\text{tm}} \langle P^+(x) \hat{P}^-(0) \rangle + \delta_{\text{PCVC}} \langle \hat{P}^-(0) \rangle , \quad (3.4)$$

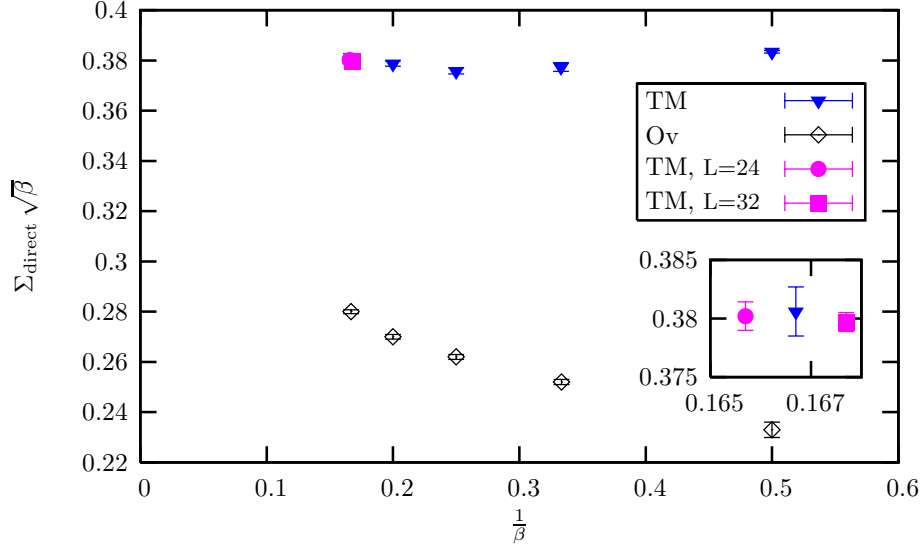


Figure 2: Comparison of the scalar condensate using OV and TM fermions as a function of $1/\beta$. In the inset data on larger lattices are shown, demonstrating the smallness of finite size effects.

where δ_{PCVC} is the chiral rotation in the twisted mass formulation and

$$\hat{P}^{\pm}(y) = \frac{1}{d} \sum_{\mu=1}^d \frac{1}{2} \left[\bar{\psi}(y) \gamma_5 \tau_{\pm} U_{\mu}(y) \psi(y + \hat{\mu}) + \bar{\psi}(y) \gamma_5 \tau_{\pm} U_{\mu}^{\dagger}(y - \hat{\mu}) \psi(y - \hat{\mu}) \right]. \quad (3.5)$$

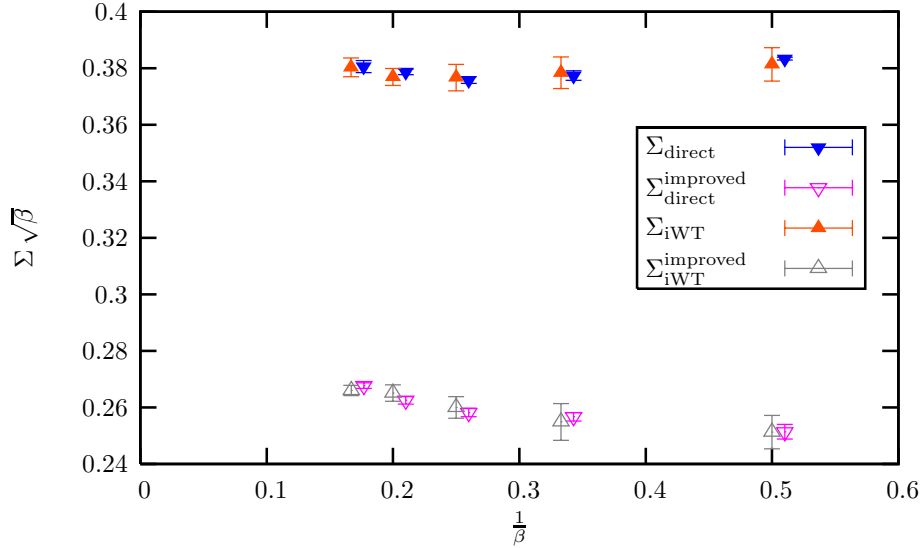


Figure 3: The scalar condensate for TM fermions as a function of $1/\beta$, using both, un-improved and improved currents.

Fig. 3 shows the comparison using the improved currents in eqs. (3.3, 3.5) with the un-improved currents in eqs. (2.3, 2.4). We find that employing the improved current, the values of the scalar condensate from TM and OV fermions approach each other. The effect of the improved current is illustrated in fig. 4, which shows the correlator of the pseudo-scalar meson using

the un-improved and the improved currents for the case of TM fermions. It can be clearly observed that the short distance behaviour is significantly altered leading to the different behaviour of the scalar condensate.

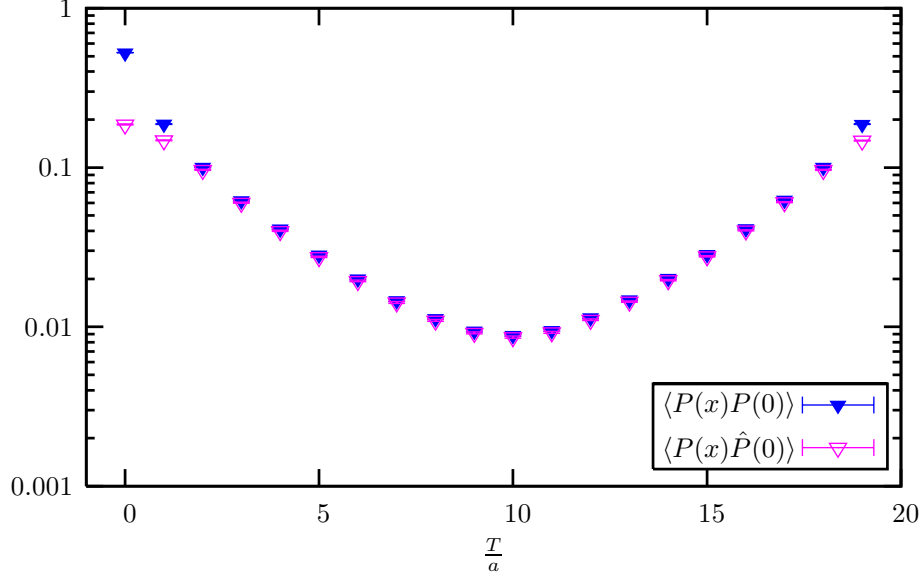


Figure 4: The correlator $\langle P(t)\hat{P}(0) \rangle$ and $\langle P(t)P(0) \rangle$ for twisted mass fermions.

In fig. 5, $\sqrt{\beta}\Sigma_{\text{sub}}$ of eq. (3.2) is plotted as a function of $1/\beta$. From this figure, we see that $\sqrt{\beta}\Sigma_{\text{sub}}$ shows the expected $O(a^2)$ scaling. In addition, the results for TM fermions using the improved current is very close to the one obtained from OV fermions. However, the continuum limit values of the scalar condensate are still not fully consistent.

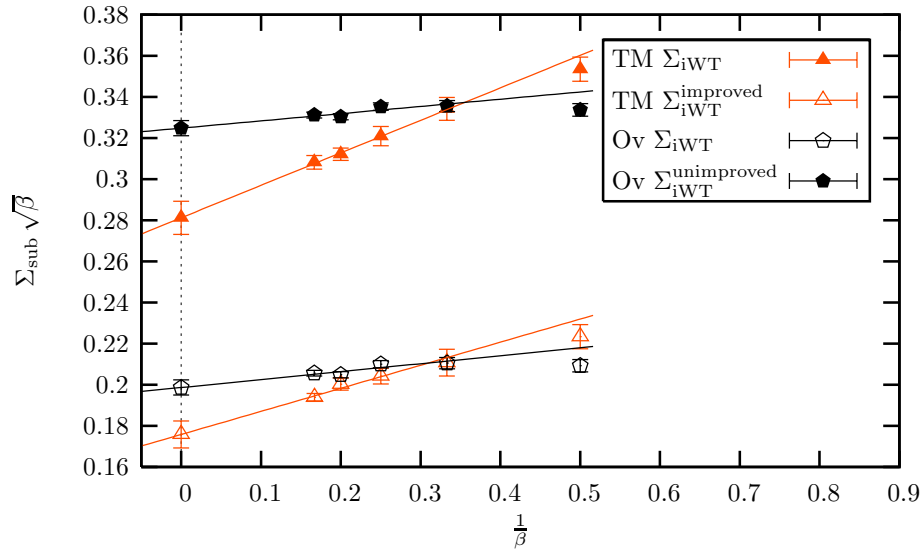


Figure 5: The subtracted condensate for fixed $z = (m_q \sqrt{\beta})^{2/3} = 0.4$

4. Summary

We have performed numerical simulations in the 2-flavour Schwinger model thought as a test ground for lattice QCD. In our work, the re-weighting, the HMC algorithm for the calculation of the fermion determinant and the spectral representation for the correlation function are used.

We carried out a scaling test of the scalar condensate for OV and TM fermions. Quite unexpectedly, we found a large discrepancy for the values of the scalar condensate between TM and OV fermions, even in the continuum limit. The discrepancy is particularly significant when an un-improved (local) current is used for TM fermions. We have suggested to use a 1-point splitting current which indeed improves the situation but does not seem to remove the discrepancy completely.

As a curious observation, we remark that we have also computed the scalar condensate by using an un-improved definition for OV fermions defined by $\tilde{P}(x) = \bar{\psi}(x)\gamma_5\psi(x)$, in which there is no factor $(1 - \frac{aD_0}{2})$. In this case, the value for the scalar condensate is almost the same value as the one obtained from the un-improved current in the case of TM fermions.

The reasons for the quite surprising outcomes of our investigation are presently explored. Clearly, if our findings are not a specialty of the 2-dimensional Schwinger model but generalize to lattice QCD, this would have serious implications for the determination of the scalar condensate in QCD.

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